13-4 Law of Sines

Main Ideas

- Solve problems by using the Law of Sines.
- Determine whether a triangle has one, two, or no solutions.

New Vocabulary

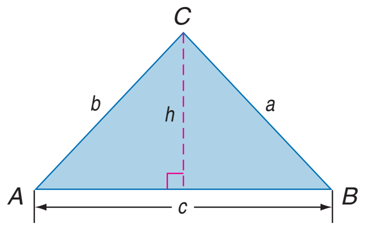
Law of Sines

GET READY for the Lesson

You know how to find the area of a triangle when the base and the height are known. Using this formula, the area of $\triangle ABC$ below is $\frac{1}{2}ch$. If the height h of this triangle were not known, you could still find the area given the measures of angle A and the length of side b.

$$\sin A = \frac{h}{b} \rightarrow h = b \sin A$$

By combining this equation with the area formula, you can find a new formula for the area of the triangle.



$$\text{Area} = \frac{1}{2}ch \rightarrow \text{Area} = \frac{1}{2}c(b \sin A)$$

Law of Sines You can find two other formulas for the area of the triangle above in a similar way.

Study Tip

Area Formulas

These formulas allow you to find the area of any triangle when you know the measures of two sides and the included angle.

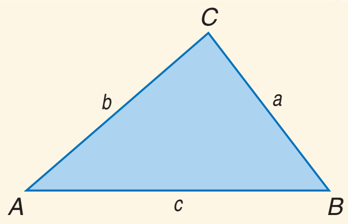
KEY CONCEPT — *Area of a Triangle*

Words The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.

Symbols area $= \frac{1}{2}bc \sin A$

area $= \frac{1}{2}ac \sin B$

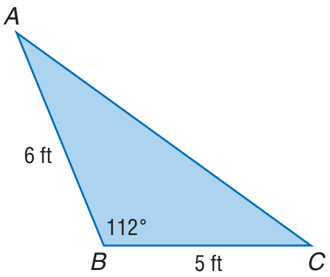
area $= \frac{1}{2}ab \sin C$



EXAMPLE 1 Find the Area of a Triangle

Find the area of $\triangle ABC$ to the nearest tenth.

In this triangle, $a = 5$, $c = 6$, and $B = 112°$. Choose the second formula because you know the values of its variables.



Area $= \frac{1}{2}ac \sin B$	Area formula
$= \frac{1}{2}(5)(6) \sin 112°$	Replace a with 5, c with 6, and B with 112°.
≈ 13.9	To the nearest tenth, the area is 13.9 square feet.

CHECK Your Progress

1. Find the area of $\triangle ABC$ to the nearest tenth if $A = 31°$, $b = 18$ m, and $c = 22$ m.

All of the area formulas for $\triangle ABC$ represent the area of the same triangle. So, $\frac{1}{2}bc \sin A$, $\frac{1}{2}ac \sin B$, and $\frac{1}{2}ab \sin C$ are all equal. You can use this fact to derive the **Law of Sines**.

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C \quad \text{Set area formulas equal to each other.}$$

$$\frac{\frac{1}{2}bc \sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac \sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab \sin C}{\frac{1}{2}abc} \quad \text{Divide each expression by } \tfrac{1}{2}abc.$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{Simplify.}$$

Study Tip

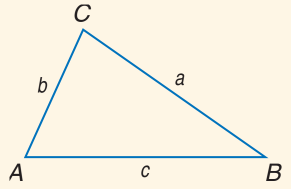
Alternate Representations

The Law of Sines may also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

KEY CONCEPT — ***Law of Sines***

Let $\triangle ABC$ be any triangle with a, b, and c representing the measures of sides opposite angles with measurements A, B, and C respectively. Then,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$



The Law of Sines can be used to write three different equations.

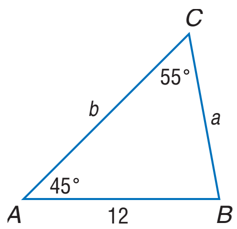
$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{or} \quad \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin C}{c}$$

In Lesson 13-1, you learned how to solve right triangles. To solve *any* triangle, you can apply the Law of Sines if you know

- the measures of two angles and any side or
- the measures of two sides and the angle opposite one of them.

EXAMPLE Solve a Triangle Given Two Angles and a Side

2 **Solve $\triangle ABC$.**



You are given the measures of two angles and a side. First, find the measure of the third angle.

$45° + 55° + B = 180°$ — The sum of the angle measures of a triangle is 180°.

$B = 80°$ — $180 - (45 + 55) = 80$

Now use the Law of Sines to find a and b. Write two equations, each with one variable.

$\frac{\sin A}{a} = \frac{\sin C}{c}$	Law of Sines	$\frac{\sin B}{b} = \frac{\sin C}{c}$
$\frac{\sin 45°}{a} = \frac{\sin 55°}{12}$	Replace A with 45°, B with 80°, C with 55°, and c with 12.	$\frac{\sin 80°}{b} = \frac{\sin 55°}{12}$
$a = \frac{12 \sin 45°}{\sin 55°}$	Solve for the variable.	$b = \frac{12 \sin 80°}{\sin 55°}$
$a \approx 10.4$	Use a calculator.	$b \approx 14.4$

Therefore, $B = 80°$, $a \approx 10.4$, and $b \approx 14.4$.

CHECK Your Progress

2. Solve $\triangle FGH$ if $m\angle G = 80°$, $m\angle H = 40°$, and $g = 14$.

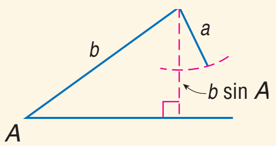
One, Two, or No Solutions When solving a triangle, you must analyze the data you are given to determine whether there is a solution. For example, if you are given the measures of two angles and a side, as in Example 2, the triangle has a unique solution. However, if you are given the measures of two sides and the angle opposite one of them, a single solution may not exist. One of the following will be true.

- No triangle exists, and there is no solution.
- Exactly one triangle exists, and there is one solution.
- Two triangles exist, and there are two solutions.

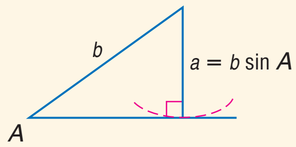
KEY CONCEPT *Possible Triangles Given Two Sides and One Opposite Angle*

Suppose you are given *a*, *b*, and *A* for a triangle.

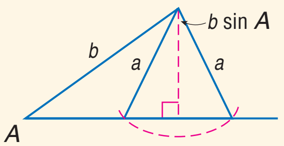
A Is Acute ($A < 90°$).



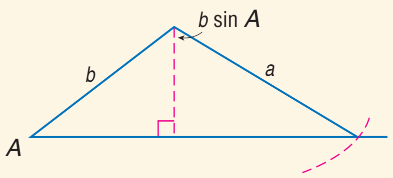
$a < b \sin A$
no solution



$a = b \sin A$
one solution

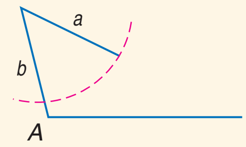


$b > a > b \sin A$
two solutions

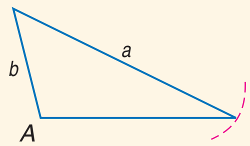


$a \geq b$
one solution

A Is Right or Obtuse ($A \geq 90°$).



$a \leq b$
no solution



$a > b$
one solution

EXAMPLE 3 **One Solution**

In $\triangle ABC$, $A = 118°$, $a = 20$, and $b = 17$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Because angle A is obtuse and $a > b$, you know that one solution exists.

Use the Law of Sines to find B.

$\frac{\sin B}{17} = \frac{\sin 118°}{20}$ Law of Sines

$\sin B = \frac{17 \sin 118°}{20}$ Multiply each side by 17.

$\sin B \approx 0.7505$ Use a calculator.

$B \approx 49°$ Use the $\sin^{-1}$ function.

The measure of angle C is approximately $180 - (118 + 49)$ or $13°$.

Use the Law of Sines again to find c.

$\frac{\sin 13}{c} = \frac{\sin 118°}{20}$ Law of Sines

$c = \frac{20 \sin 13°}{\sin 118°}$ or about 5.1

Therefore, $B \approx 49°$, $C \approx 13°$, and $c \approx 5.1$.

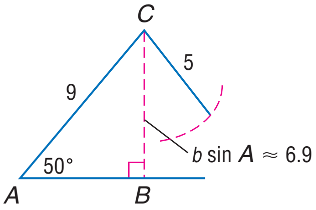
CHECK Your Progress

3. In $\triangle ABC$, $B = 95°$, $b = 19$, and $c = 12$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

EXAMPLE No Solution

4 In $\triangle ABC$, $A = 50°$, $a = 5$, and $b = 9$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

Since angle A is acute, find $b \sin A$ and compare it with a.



$b \sin A = 9 \sin 50°$ Replace b with 9 and A with 50°.

≈ 6.9 Use a calculator.

Since $5 < 6.9$, there is no solution.

Study Tip

A Is Acute

We compare $b \sin A$ to a because $b \sin A$ is the minimum distance from C to $\overline{AB}$ when A is acute.

CHECK Your Progress

4. In $\triangle ABC$, $B = 95°$, $b = 10$, and $c = 12$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

When two solutions for a triangle exist, it is called the *ambiguous case.*

EXAMPLE Two Solutions

5 In $\triangle ABC$, $A = 39°$, $a = 10$, and $b = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.

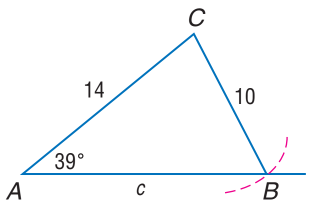
Since angle A is acute, find $b \sin A$ and compare it with a.

$b \sin A = 14 \sin 39°$ Replace b with 14 and A with 39°.

≈ 8.81 Use a calculator.

Since $14 > 10 > 8.81$, there are two solutions. Thus, there are two possible triangles to be solved.

Case 1 Acute Angle B



First, use the Law of Sines to find B.

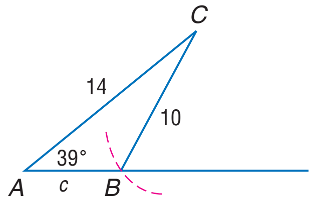
$$\frac{\sin B}{14} = \frac{\sin 39°}{10}$$

$$\sin B = \frac{14 \sin 39°}{10}$$

$$\sin B = 0.8810$$

$$B \approx 62°$$

Case 2 Obtuse Angle B



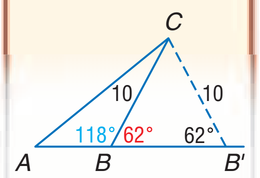
To find B, you need to find an obtuse angle whose sine is also 0.8810. To do this, subtract the angle given by your calculator, 62°, from 180°. So B is approximately $180 - 62$ or 118°.

The measure of angle C is approximately $180 - (39 + 118)$ or 23°.

Study Tip

Alternate Method

Another way to find the obtuse angle in Case 2 of Example 5 is to notice in the figure below that $\triangle CBB'$ is isosceles. Since the base angles of an isosceles triangle are always congruent and $m\angle B' = 62°$, $m\angle CBB' = 62°$. Also, $\angle ABC$ and $m\angle CBB'$ are supplementary. Therefore, $m\angle ABC = 180° - 62°$ or 118°.



The measure of angle C is approximately $180 - (39 + 62)$ or $79°$.

$$\frac{\sin 79°}{c} = \frac{\sin 39°}{10}$$
$$c = \frac{10 \sin 79°}{\sin 39°}$$
$$c \approx 15.6$$

Therefore, $B \approx 62°$, $C \approx 79°$, and $c \approx 15.6$.

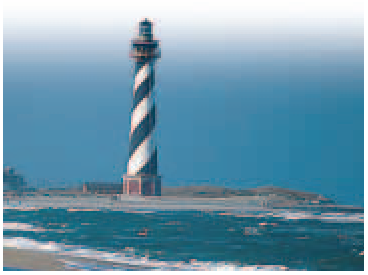
Use the Law of Sines to find c.

$$\frac{\sin 23°}{c} = \frac{\sin 39°}{10}$$
$$c = \frac{10 \sin 23°}{\sin 39°}$$
$$c \approx 6.2$$

Therefore, $B \approx 118°$, $C \approx 23°$, and $c \approx 6.2$.

CHECK Your Progress

5. In $\triangle ABC$, $A = 44°$, $b = 19$, and $a = 14$. Determine whether $\triangle ABC$ has *no* solution, *one* solution, or *two* solutions. Then solve $\triangle ABC$.



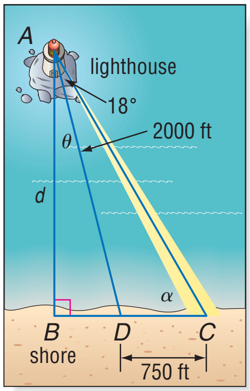
Real-World Link

Standing 208 feet tall, the Cape Hatteras Lighthouse in North Carolina is the tallest lighthouse in the United States.

Source: www.oldcapehatteraslighthouse.com

Real-World EXAMPLE — Use the Law of Sines to Solve a Problem

6 **LIGHTHOUSES** **The light on a lighthouse revolves counterclockwise at a steady rate of one revolution per minute. The beam strikes a point on the shore that is 2000 feet from the lighthouse. Three seconds later, the light strikes a point 750 feet further down the shore. To the nearest foot, how far is the lighthouse from the shore?**



Because the lighthouse makes one revolution every 60 seconds, the angle through which the light revolves in 3 seconds is $\frac{3}{60}(360°)$ or $18°$.

Use the Law of Sines to find the measure of angle α.

$\frac{\sin \alpha}{2000} = \frac{\sin 18°}{750}$	Law of Sines
$\sin \alpha = \frac{2000 \sin 18°}{750}$	Multiply each side by 2000.
$\sin \alpha \approx 0.8240$	Use a calculator.
$\alpha \approx 55°$	Use the $\sin^{-1}$ function.

Use this angle measure to find the measure of angle θ.

$\alpha + m\angle BAC = 90°$	Angles α and $\angle BAC$ are complementary.
$55° + (\theta + 18°) \approx 90°$	$\alpha < 55°$ and $m\angle BAC = \theta + 18°$
$\theta \approx 17°$	Solve for θ.

To find the distance from the lighthouse to the shore, solve $\triangle ABD$ for d.

$\cos \theta = \frac{AB}{AD}$	Cosine ratio
$\cos 17° \approx \frac{d}{2000}$	$\theta = 17°$ and $AD = 2000$
$d \approx 2000 \cos 17°$	Solve for d.
$d \approx 1913$	Use a calculator.

To the nearest foot, it is 1913 feet from the lighthouse to the shore.

CHECK Your Progress

6. The beam of light from another lighthouse strikes the shore 3000 feet away. Three seconds later, the beam strikes 1200 feet farther down the shore. To the nearest foot, how far is this lighthouse from the shore?

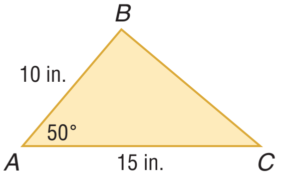
Online Personal Tutor at algebra2.com

CHECK Your Understanding

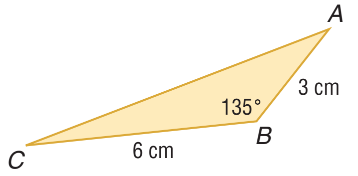
Example 1 (p. 785)

Find the area of $\triangle ABC$ to the nearest tenth.

1.



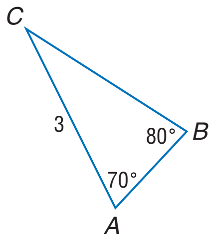
2.



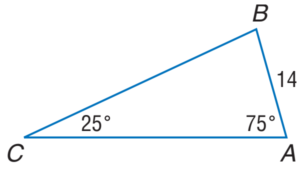
Example 2 (pp. 786–787)

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

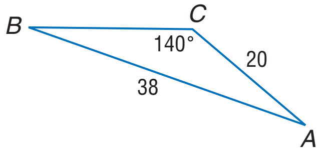
3.



4.



5.



Examples 3–5 (pp. 787–789)

Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

6. $A = 123°, a = 12, b = 23$

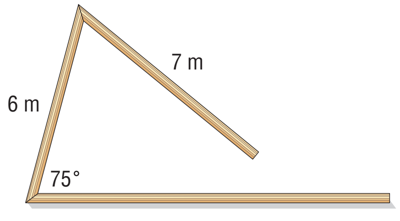
7. $A = 30°, a = 3, b = 4$

8. $A = 55°, a = 10, b = 5$

9. $A = 145°, a = 18, b = 10$

Example 6 (p. 789)

10. WOODWORKING Latisha is to join a 6-meter beam to a 7-meter beam so the angle opposite the 7-meter beam measures 75°. To what length should Latisha cut the third beam in order to form a triangular brace? Round to the nearest tenth.



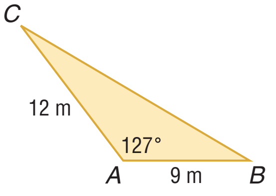
Exercises

HOMEWORK HELP

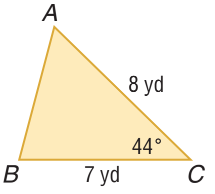
For Exercises	See Examples
11–16	1
17–30	2–5
31, 32	6

Find the area of $\triangle ABC$ to the nearest tenth.

11.



12.



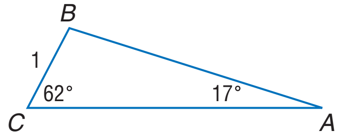
13. $B = 85°, c = 23$ ft, $a = 50$ ft

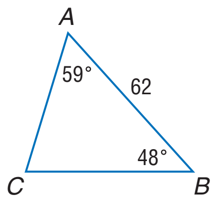
14. $A = 60°, b = 12$ cm, $c = 12$ cm

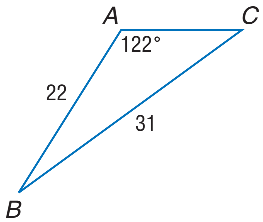
15. $C = 136°, a = 3$ m, $b = 4$ m

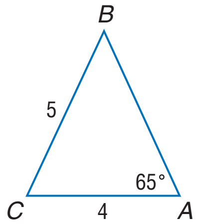
16. $B = 32°, a = 11$ mi, $c = 5$ mi

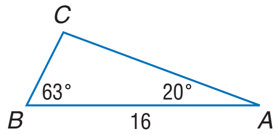
Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

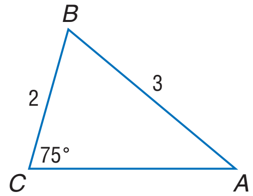
17.


18.


19.


20.


21.


22.


Determine whether each triangle has *no* solution, *one* solution, or *two* solutions. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

23. $A = 124°, a = 1, b = 2$

24. $A = 99°, a = 2.5, b = 1.5$

25. $A = 33°, a = 2, b = 3.5$

26. $A = 68°, a = 3, b = 5$

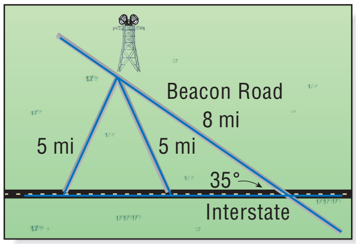
27. $A = 30°, a = 14, b = 28$

28. $A = 61°, a = 23, b = 8$

29. $A = 52°, a = 190, b = 200$

30. $A = 80°, a = 9, b = 9.1$

31. RADIO A radio station providing local tourist information has its transmitter on Beacon Road, 8 miles from where it intersects with the interstate highway. If the radio station has a range of 5 miles, between what two distances from the intersection can cars on the interstate tune in to hear this information?



32. FORESTRY Two forest rangers, 12 miles from each other on a straight service road, both sight an illegal bonfire away from the road. Using their radios to communicate with each other, they determine that the fire is between them. The first ranger's line of sight to the fire makes an angle of 38° with the road, and the second ranger's line of sight to the fire makes a 63° angle with the road. How far is the fire from each ranger?

Real-World Link

Hot-air balloons range in size from approximately 54,000 cubic feet to over 250,000 cubic feet.

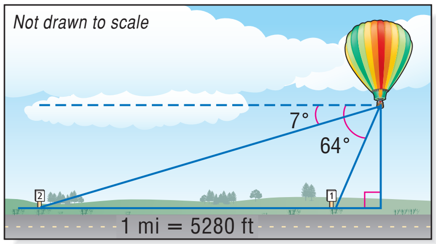
Source: www.unicorn-ballon.com

Solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree.

33. $A = 50°, a = 2.5, c = 3$

34. $B = 18°, C = 142°, b = 20$

35. BALLOONING As a hot-air balloon crosses over a straight portion of interstate highway, its pilot eyes two consecutive mileposts on the same side of the balloon. When viewing the mileposts the angles of depression are 64° and 7°. How high is the balloon to the nearest foot?



EXTRA PRACTICE
See pages 921, 938.

Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

36. **OPEN ENDED** Give an example of a triangle that has two solutions by listing measures for A, a, and b, where a and b are in centimeters. Then draw both cases using a ruler and protractor.

37. **FIND THE ERROR** Dulce and Gabe are finding the area of $\triangle ABC$ for $A = 64°$, $a = 15$ meters, and $b = 8$ meters using the sine function. Who is correct? Explain your reasoning.

Dulce	Gabe
$\text{Area} = \frac{1}{2}(15)(8)\sin 64°$	$\text{Area} = \frac{1}{2}(15)(8)\sin 87.4°$
$\approx 53.9\ m^2$	$\approx 59.9\ m^2$

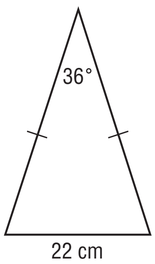
38. **REASONING** Determine whether the following statement is *sometimes*, *always* or *never* true. Explain your reasoning.

If given the measure of two sides of a triangle and the angle opposite one of them, you will be able to find a unique solution.

39. ***Writing in Math*** Use the information on page 785 to explain how trigonometry can be used to find the area of a triangle.

STANDARDIZED TEST PRACTICE

40. **ACT/SAT** Which of the following is the perimeter of the triangle shown?



A 49.0 cm
B 66.0 cm
C 91.4 cm
D 93.2 cm

41. **REVIEW** The longest side of a triangle is 67 inches. Two angles have measures of 47° and 55°. What is the length of the shortest leg of the triangle?

F 50.1 in.
G 56.1 in.
H 60.1 in.
J 62.3 in.

Spiral Review

Find the exact value of each trigonometric function. (Lesson 13-3)

42. $\cos 30°$

43. $\cot \left(\frac{\pi}{3}\right)$

44. $\csc \left(\frac{\pi}{4}\right)$

Find one angle with positive measure and one angle with negative measure coterminal with each angle. (Lesson 13-2)

45. 300°

46. 47°

47. $\frac{5\pi}{3}$

48. **AERONAUTICS** A rocket rises 20 feet in the first second, 60 feet in the second second, and 100 feet in the third second. If it continues at this rate, how many feet will it rise in the 20th second? (Lesson 11-1)

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.** (Lesson 13-1)

49. $a^2 = 3^2 + 5^2 - 2(3)(5)\cos 85°$

50. $c^2 = 12^2 + 10^2 - 2(12)(10)\cos 40°$

51. $7^2 = 11^2 + 9^2 - 2(11)(9)\cos B°$

52. $13^2 = 8^2 + 6^2 - 2(8)(6)\cos A°$